

ON AN ENDOMORPHISM RING OF LOCAL COHOMOLOGY

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ABSTRACT. Let I be an ideal of a local ring (R, \mathfrak{m}) with $d = \dim R$. For the local cohomology module $H_I^i(R)$ it is a well-known fact that it vanishes for $i > d$ and is an Artinian R -module for $i = d$. In the case that the Hartshorne-Lichtenbaum Vanishing Theorem fails, that is $H_I^d(R) \neq 0$, we explore its fine structure. In particular, we investigate its endomorphism ring and related connectedness properties. In the case R is complete we prove - as a technical tool - that $H_I^d(R) \simeq H_{\mathfrak{m}}^d(R/J)$ for a certain ideal $J \subset R$. Thus, properties of $H_I^d(R)$ and its Matlis dual might be described in terms of the local cohomology supported in the maximal ideal.

1. INTRODUCTION

Let $I \subset R$ denote an ideal of a local ring (R, \mathfrak{m}) . Let M be a finitely generated R -module with $d = \dim M$. For an integer $i \in \mathbb{Z}$ let $H_I^i(M)$ denote the i -th local cohomology module of M with respect to I as introduced by Grothendieck (see [5] and [3]). Of a particular interest are the first non-vanishing (resp. the last non-vanishing) cohomological degree of the local cohomology modules $H_I^i(M)$, known as the grade $\text{grade}(I, M)$ (resp. cohomological dimension $\text{cd}(I, M)$). It is a well-known fact that

$$\text{grade}(I, M) \leq \text{cd}(I, M) \leq \dim M.$$

In the case of $I = \mathfrak{m}$ it follows that $\text{cd}(\mathfrak{m}, M) = \dim M$ (see [5]). While for an arbitrary ideal $I \subset R$ the Hartshorne-Lichtenbaum Vanishing Theorem says that the following conditions are equivalent:

- (1) $H_I^d(M) = 0$.
- (2) $\dim \hat{R}/I\hat{R} + \mathfrak{p} > 0$ for all $\mathfrak{p} \in \text{Ass}_{\hat{R}} \hat{M}$ such that $\dim \hat{R}/\mathfrak{p} = d$.

(see [7] and [3]). Here \hat{M} resp. \hat{R} denotes the completion of M resp. R . Moreover, it follows that $H_I^d(M)$ is an Artinian R -module (see [7] and [13]). Furthermore, there is an explicit description of the Artinian R -module $H_I^d(M)$ by its secondary decomposition and its attached prime ideals (see [4, Section 3]). Let $E_R(R/\mathfrak{m})$ denote the injective hull of the residue field. Then $\text{Hom}_R(H_I^d(M), E_R(R/\mathfrak{m}))$ is a finitely generated \hat{R} -module. One of our interest is to study the properties of it.

In recent research there is an interest in endomorphism rings of certain local cohomology modules $H_I^i(R)$. This was done in the case of $i = \dim R$ and $I = \mathfrak{m}$ by Hochster and Huneke (see [8]) and in the case of $i = \text{height } I$ and R a Gorenstein ring (see [11] and the references there). Here we continue with the case of $i = \dim R$ and an arbitrary ideal $I \subset R$. In particular we investigate the natural ring homomorphism

$$\hat{R} \rightarrow \text{Hom}_{\hat{R}}(H_I^d(R), H_I^d(R)), \quad d = \dim R.$$

We describe its kernel, characterize when it is an isomorphism, prove that the endomorphism ring $\text{Hom}_{\hat{R}}(H_I^d(R), H_I^d(R))$ is commutative and decide when it is a local Noetherian ring. Note that in the case of $I = \mathfrak{m}$ we recover results shown by Hochster and Huneke (see [8]). In fact, we use their result in our proof.

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Theorem 1.1. *Let I denote an ideal of a complete local ring (R, \mathfrak{m}) . Let $J = \text{Ann}_R H_I^d(R)$ where $d = \dim R$.*

- (a) *The endomorphism ring $\text{Hom}_R(H_I^d(R), H_I^d(R))$ is a commutative semi-local ring, finitely generated as R -module.*
- (b) *The natural homomorphism $R/J \rightarrow \text{Hom}_R(H_I^d(R), H_I^d(R))$ is an isomorphism if and only if R/J satisfies the condition S_2 .*

Moreover, we describe the ideal J explicitly in terms of R and I . Of course this is only of an interest whenever $H_I^d(R) \neq 0$. That is, when the ideal $I \subset R$ does not satisfy condition (2) above. As an important step towards the proof of Theorem 1.1 we prove the following result:

Theorem 1.2. *Let (R, \mathfrak{m}) denote a complete local ring. Let $I \subset R$ denote an ideal. Let M be a finitely generated R -module with $d = \dim M$. Then there is an ideal $J \subset R$ such that*

$$H_I^d(M) \simeq H_{\mathfrak{m}}^d(M/JM) \simeq H_{\mathfrak{m}}^d(M)/JH_{\mathfrak{m}}^d(M).$$

The ideal J is described explicitly.

2. AUXILIARY RESULTS

Let (R, \mathfrak{m}) denote a local ring. Let $E_R(R/\mathfrak{m})$ denote the injective hull of the residue field $R/\mathfrak{m} = k$. The Matlis duality functor $\text{Hom}_R(\cdot, E_R(R/\mathfrak{m}))$ is denoted by $D(\cdot)$. We need a Lemma concerning R -modules such that $\text{Supp}_R M \subseteq \{\mathfrak{m}\}$, see e.g. [12, Lemma 2.1].

Lemma 2.1. *Let M be an R -module and N an \hat{R} -module.*

- (a) *Suppose $\text{Supp}_R M \subseteq V(\mathfrak{m})$. Then M admits a unique \hat{R} -module structure compatible with its R -module structure such that the natural map $M \otimes_R \hat{R} \rightarrow M$ is an isomorphism.*
- (b) *The module $\text{Ext}_R^i(M, N)$, $i \in \mathbb{Z}$, might be considered as an \hat{R} -module such that there is a natural isomorphism $\text{Ext}_R^i(M, N) \simeq \text{Ext}_{\hat{R}}^i(M \otimes_R \hat{R}, N)$.*
- (c) *The Matlis dual $D(M)$ admits a natural \hat{R} -module structure.*

We need the definition of the canonical module of a finitely generated R -module M . To this end let (R, \mathfrak{m}) be the epimorphic image of a local Gorenstein ring (S, \mathfrak{n}) and $n = \dim S$.

Definition 2.2. (A) For a finitely generated R -module M we consider

$$K(M) = \text{Hom}_R(H_{\mathfrak{m}}^d(M), E_R(R/\mathfrak{m})), \quad d = \dim M.$$

Because $H_{\mathfrak{m}}^d(M)$ is an Artinian R -module it turns out that $K(M)$ is a finitely generated \hat{R} -module. (B) This is closely related to the notion of the canonical module K_M of a finitely generated R -module M . To this end we have to assume that (R, \mathfrak{m}) is the epimorphic image of a local Gorenstein ring (S, \mathfrak{n}) with $n = \dim S$. Then $K_M = \text{Ext}_S^{n-d}(M, S)$ is called the canonical module of M . By virtue of the Cohen Structure Theorem and (A) there is an isomorphism $K(M) \simeq K_{\hat{M}}$. By the Local Duality Theorem (see [5]) there are the following isomorphisms

$$\text{Hom}_R(H_{\mathfrak{m}}^d(M), E_R(R/\mathfrak{m})) \simeq K(M) \simeq K_{\hat{M}}.$$

Moreover, in order to describe certain results on the canonical module we need a further technical definition.

Definition 2.3. For an ideal $I \subset R$ with $\dim R/I = d$ we will denote by I_d the intersection of those primary components in a minimal reduced primary decomposition of I which are of dimension d . If $Z \subset \text{Spec } R$ and $d \in \mathbb{N}$, then we put $Z_d = \{\mathfrak{p} \in Z \mid \dim R/\mathfrak{p} = d\}$.

In the following we summarize a few results about the canonical module K_M as well as of $K(M)$. Most of the statements are known in the literature. When we write K_M we always assume that R is a factor ring of a Gorenstein ring S as above.

Proposition 2.4. *Let M denote a finitely generated R -module. With the previous conventions and notation the following statements hold:*

- (a) $\text{Ass}_{\hat{R}} K(M) = (\text{Ass}_{\hat{R}} \hat{M})_d$ and $\text{Ass}_R K_M = (\text{Ass}_R M)_d$.
- (b) $\text{Ann}_{\hat{R}} K(M) = (\text{Ann}_{\hat{R}} \hat{M})_d$ and $\text{Ann}_R K_M = (\text{Ann}_R M)_d$.
- (c) *The module K_M satisfies Serre's condition S_2 , that is*

$$\text{depth}(K_M)_{\mathfrak{p}} \geq \min\{2, \dim(K_M)_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \text{Supp}_R K_M.$$

- (d) *The kernel of the natural map $R \rightarrow \text{Hom}_R(K_R, K_R)$ is 0_d . It is an isomorphism if and only if R satisfies S_2 .*

Proof. For the proof of (a) and (b) we refer to [5, Proposition 6.6]. For the proof of (c) see [8, Lemma 1.9] and for (d) see [1] and [2, Proposition 1.2]. For some additional information we refer also to [9, Lemma 1.9]. \square

3. REMARKS TO THE HARTSHORNE-LICHTENBAUM VANISHING THEOREM

In this section let M denote a d -dimensional, finitely generated R -module. Here (R, \mathfrak{m}) is a local ring. The functor $\hat{}$ denotes the completion functor.

For an R -module M let $0 = \cap_{i=1}^n Q_i(M)$ denote a reduced minimal primary decomposition of the zero submodule of M . That is $M/Q_i(M), i = 1, \dots, n$, is a \mathfrak{p}_i -primary R -module. Clearly $\text{Ass}_R M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

Definition 3.1. Let $I \subset R$ denote an ideal of R . We define two disjoint subsets U, V of $\text{Ass}_R M$ related to I .

- (a) $U = \{\mathfrak{p} \in \text{Ass}_R M \mid \dim R/\mathfrak{p} = d \text{ and } \dim R/I + \mathfrak{p} = 0\}$.
- (b) $V = \{\mathfrak{p} \in \text{Ass}_R M \mid \dim R/\mathfrak{p} < d \text{ or } \dim R/\mathfrak{p} = d \text{ and } \dim R/I + \mathfrak{p} > 0\}$.

Finally we define $Q_I(M) = \cap_{\mathfrak{p}_i \in U} Q_i(M)$. In the case $U = \emptyset$ put $Q_I(M) = M$.

The following Lemma gives a better understanding of the previous definitions.

Lemma 3.2. *With the previous notation we have that*

$$\text{Ass}_R Q_I(M) = V, \text{Ass}_R M/Q_I(M) = U \text{ and } U \cup V = \text{Ass}_R M.$$

Proof. The proof is an easy consequence of the primary decomposition of $M, M/Q_I(M)$ and $Q_I(M)$ (see [10, Lemma 2.7]). \square

Now we are prepared in order to establish the first main result of this section. It explains in more detail the structure of $H_I^d(M), d = \dim M$.

Theorem 3.3. *Let I denote an ideal of a local ring (R, \mathfrak{m}) . Let M be a finitely generated R -module and $d = \dim M$. Then there is a natural isomorphism*

$$H_I^d(M) \simeq H_{\mathfrak{m}\hat{R}}^d(\hat{M}/Q_{I\hat{R}}(\hat{M})),$$

where $\hat{}$ denotes the \mathfrak{m} -adic completion.

Proof. First note that $H_I^d(M)$ is an Artinian R -module. So it admits a unique \hat{R} -module structure compatible with its R -module structure such that the natural homomorphism

$$H_{I\hat{R}}^d(\hat{M}) \simeq H_I^d(M) \otimes_R \hat{R} \rightarrow H_I^d(M)$$

is an isomorphism. That is, without loss of generality we may assume that R is complete.

Now apply the local cohomology functor to the short exact sequence

$$0 \rightarrow Q_I(M) \rightarrow M \rightarrow M/Q_I(M) \rightarrow 0$$

it implies a natural isomorphism $H_I^d(M) \simeq H_I^d(M/Q_I(M))$. To this end recall that $H_I^i(Q_I(M)) = 0$ for all $i \geq d$. The vanishing for $i = d$ follows by the Hartshorne-Lichtenbaum Vanishing Theorem because of $\text{Ass}_R Q = V$, where $Q = Q_I(M)$. By the base change of local cohomology there is the isomorphism

$$H_I^d(M/Q_I(M)) \simeq H_{I+\text{Ann}_R M/Q}^d(M/Q).$$

In order to complete the proof it is enough to show that $\mathfrak{m} = \text{Rad}(I + \text{Ann}_R M/Q)$. To this end consider

$$V(I + \text{Ann}_R M/Q) = V(I) \cup \text{Supp}_R M/Q = \cup_{\mathfrak{p} \in U} V(I + \mathfrak{p}) = \{\mathfrak{m}\},$$

as required. \square

In the case of $M = R$ in Theorem 3.3 it follow that $H_I^d(R) = H_{\mathfrak{m}\hat{R}}^d(\hat{R}/Q_{I\hat{R}}(\hat{R}))$. By the definition $Q_{I\hat{R}}(\hat{R})$ is equal to the intersection of all the \mathfrak{p} -primary component of a reduced minimal primary decomposition of the zero ideal in \hat{R} such that $\dim \hat{R}/\mathfrak{p} = \dim R$ and $\dim \hat{R}/I\hat{R} + \mathfrak{p} = 0$. Next we want to extend this to the case of an R -module M .

Definition 3.4. Let M denote a finitely generated module over the local ring (R, \mathfrak{m}) . Let $I \subset R$ denote an ideal. Then define $P_I(M)$ as the intersection of all the primary components of $\text{Ann}_R M$ such that $\dim R/\mathfrak{p} = \dim M$ and $\dim R/I + \mathfrak{p} = 0$. Clearly $P_I(M)$ is the preimage of $Q_{I\hat{R}/\text{Ann}_R M}(\hat{R}/\text{Ann}_R M)$ in R .

With these preparations we are able to prove the extension we have in mind.

Corollary 3.5. Let M denote a finitely generated R -module and $d = \dim M$. Let $I \subset R$ be an ideal. Then

$$H_I^d(M) \simeq H_{\mathfrak{m}\hat{R}}^d(\hat{M}/P_I(\hat{M})\hat{M}),$$

where $P_I(\hat{M}) \subset \hat{R}$ is the ideal as defined in Definition 3.4.

Proof. As in the beginning of proof of Theorem 3.3 we may assume that R is a complete local ring without loss of generality. Let $\bar{R} = R/\text{Ann}_R M$. Then by base change and the right exactness there are the isomorphisms

$$H_I^d(M) \simeq H_{I\bar{R}}^d(M) \simeq H_{I\bar{R}}^d(\bar{R}) \otimes_R M.$$

Now by virtue of Theorem 3.3 there is the isomorphism $H_{I\bar{R}}^d(\bar{R}) \simeq H_{\mathfrak{m}}^d(R/P_I(M))$. Therefore it follows that

$$H_{I\bar{R}}^d(\bar{R}) \otimes_R M \simeq H_{\mathfrak{m}}^d(R/P_I(M)) \otimes_R M \simeq H_{\mathfrak{m}}^d(M/P_I(M)M),$$

which finishes the proof of the statement. \square

Let A denote an Artinian R -module. Then the decreasing sequence of submodules $\{\mathfrak{m}^n A\}_{n \in \mathbb{N}}$ becomes stable. Let $\langle \mathfrak{m} \rangle A$ denote the ultimative stable value of this sequence of decreasing submodules.

Remark 3.6. Let $I \subset R$ denote an ideal. For a finitely generated R -module M there is a natural epimorphism

$$H_{\mathfrak{m}}^d(M) \rightarrow H_I^d(M) \rightarrow 0, \quad d = \dim M,$$

(see [4]). In fact (see [4, Theorem 1.1]) it induces an isomorphism

$$H_I^d(M) \simeq H_{\mathfrak{m}}^d(M) / \sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_{H_{\mathfrak{m}}^d(M)} I^n).$$

Thus the kernel is described as $\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_{H_{\mathfrak{m}}^d(M)} I^n)$.

Let us consider the previous epimorphism as an epimorphism of \hat{R} -modules. Then by Corollary 3.5 its kernel is equal to $P_I(\hat{M})H_{\mathfrak{m}\hat{R}}^d(\hat{M})$, or in other words

$$H_I^d(M) \simeq H_{\mathfrak{m}\hat{R}}^d(\hat{M})/P_I(\hat{M})H_{\mathfrak{m}\hat{R}}^d(\hat{M}).$$

This follows easily since $H_{\mathfrak{m}\hat{R}}^d(\hat{M}/P_I(\hat{M})\hat{M}) \simeq H_{\mathfrak{m}\hat{R}}^d(\hat{M}) \otimes_{\hat{R}} \hat{R}/P_I(\hat{M})$.

By virtue of Corollary 3.5 and Remark 3.6 this proves Theorem 1.2.

4. ON THE ENDOMORPHISM RING

In this Section let (R, \mathfrak{m}) denote a d -dimensional local ring. For an ideal $I \subset R$ we investigate the endomorphism ring of $H_I^d(R)$. It is non-zero if and only if $H_I^d(R)$ fails the Hartshorne-Lichtenbaum Vanishing Theorem. That is, we study the natural homomorphism

$$R \rightarrow \operatorname{Hom}_R(H_I^d(R), H_I^d(R)), \quad r \mapsto m_r,$$

where m_r denotes the multiplication map by $r \in R$. Since $H_I^d(R)$ admits the structure of an \hat{R} -module (see 2.1) it follows that $\operatorname{Hom}_R(H_I^d(R), H_I^d(R))$ has a unique natural \hat{R} -module such that the diagram

$$\begin{array}{ccc} R & \rightarrow & \operatorname{Hom}_R(H_I^d(R), H_I^d(R)) \\ \downarrow & & \parallel \\ \hat{R} & \rightarrow & \operatorname{Hom}_{\hat{R}}(H_I^d(R), H_I^d(R)). \end{array}$$

is commutative. That is, the map $R \rightarrow \operatorname{Hom}_R(H_I^d(R), H_I^d(R))$ factors through \hat{R} . Before we study the endomorphism ring we need an auxiliary statement on the Matlis dual of $H_I^d(R)$.

Lemma 4.1. *Let I denote an ideal in a local ring (R, \mathfrak{m}) and $d = \dim R$.*

- (a) $T_I(R) = \operatorname{Hom}_R(H_I^d(R), E_R(R/\mathfrak{m}))$ is a finitely generated \hat{R} -module.
- (b) $\operatorname{Ass}_{\hat{R}} T_I(R) = \{\mathfrak{p} \in \operatorname{Ass} \hat{R} \mid \dim \hat{R}/\mathfrak{p} = \dim R \text{ and } \dim \hat{R}/I\hat{R} + \mathfrak{p} = 0\}$.
- (c) $K_{\hat{R}}(\hat{R}/Q_I(\hat{R})) \simeq T_I(R)$.

Proof. The statements follow by the definition and the auxiliary results above. \square

For an R -module M the natural map $R \rightarrow \operatorname{Hom}_R(M, M)$ is in general neither injective nor surjective. For the local cohomology module $H_I^d(R)$ we get a more precise picture.

Theorem 4.2. *Let I denote an ideal in a local ring (R, \mathfrak{m}) with $d = \dim R$. Let*

$$\Phi : \hat{R} \rightarrow \operatorname{Hom}_{\hat{R}}(H_I^d(R), H_I^d(R))$$

the natural homomorphism, where $H_I^d(R)$ is as an Artinian R -module considered as an \hat{R} -module.

- (a) $\ker \Phi = Q_{I\hat{R}}(\hat{R})$.
- (b) Φ is surjective if and only if $\hat{R}/Q_{I\hat{R}}(\hat{R})$ satisfies S_2 .
- (c) $\operatorname{Hom}_{\hat{R}}(H_I^d(R), H_I^d(R))$ is a finitely generated \hat{R} -module.
- (d) $\operatorname{Hom}_{\hat{R}}(H_I^d(R), H_I^d(R))$ is a commutative semi-local Noetherian ring.

Proof. First note that as $H_I^d(R)$ is an Artinian R -module so $H_I^d(R) \simeq H_{I\hat{R}}^d(\hat{R})$ (see 2.1 for the detail). That is, without loss of generality we may assume that R is a complete local ring. By virtue of Theorem 3.3 there is the natural isomorphism $H_I^d(R) \simeq H_{\mathfrak{m}}^d(R/Q)$, $Q = Q_I(R)$. Then

$$K_{R/Q} \simeq D(H_{\mathfrak{m}}^d(R/Q)) \simeq \operatorname{Hom}_R(H_I^d(R), E_R(R/\mathfrak{m})).$$

Because $H_I^d(R)$ is Artinian the Matlis' duality provides an isomorphism

$$\operatorname{Hom}_R(H_I^d(R), H_I^d(R)) \simeq \operatorname{Hom}_R(K_{R/Q}, K_{R/Q}).$$

Therefore the kernel of Φ equals to $\operatorname{Ann}_R K_{R/Q} = Q_d$, which proves (a). Because the endomorphism ring of $H_I^d(R)$ is isomorphic to the endomorphism ring of the canonical module of $K_{R/Q}$ the results in (b), (c) and (d) are shown by Aoyama (see [1, Proposition 1.2]), Aoyama and Gôto (see [2, Theorem 3.2]) and Hochster and Huneke (see [8, (2.2)]). \square

Note that the ideal $J \subset R$ as considered in Theorem 1.1 (b) in the case of a complete local ring (R, \mathfrak{m}) coincides with $Q_I(R)$ in Theorem 4.2. In the next we want to relate some homological properties of $T_I(R)$ with those of the endomorphism ring $\operatorname{Hom}_R(H_I^d(R), H_I^d(R))$ resp. $\hat{R}/Q_{I\hat{R}}(\hat{R})$.

Theorem 4.3. *Let I be an ideal in (R, \mathfrak{m}) , a complete local ring and $\dim R = d$. For an integer $r \geq 2$ we have the following statements:*

- (a) *Suppose $R/Q_I(R)$ has S_2 . Then $T_I(R)$ satisfies the condition S_r if and only if $H_{\mathfrak{m}}^i(R/Q_I(R)) = 0$ for $d - r + 2 \leq i < d$.*
- (b) *$R/Q_I(R)$ satisfies the condition S_r if and only if $H_{\mathfrak{m}}^i(T_I(R)) = 0$ for $d - r + 2 \leq i < d$ and $R/Q_I(R) \simeq \text{Hom}_R(H_I^d(R), H_I^d(R))$.*

In particular, if $R/Q_I(R)$ has S_2 it is a Cohen-Macaulay ring if and only if the module $T_I(R)$ is Cohen-Macaulay.

Proof. By our conventions and definitions it follows that $T_I(R) \simeq K_{R/Q}$, where $Q = Q_I(R)$, and $R/Q \simeq \text{Hom}_R(H_I^d(R), H_I^d(R))$. Then the statement in (a) resp. in (b) follows by virtue of [9, 1.14] for $M = R/Q$ resp. $M = K_{R/Q}$. \square

5. ON CONNECTEDNESS AND INDECOMPOSABILITY

The next part of our investigations is to characterize the number of the maximal ideals of the endomorphism ring $\text{Hom}_R(H_I^d(R), H_I^d(R))$, $d = \dim R$ (see 4.2). To this end we need a few more preparations. First we recall a definition given by Hochster and Huneke (see [8, (3.4)]).

Definition 5.1. Let (R, \mathfrak{m}) denote a local ring. We denote by $\mathbb{G}(R)$ the undirected graph whose vertices are primes $\mathfrak{p} \in \text{Spec } R$ such that $\dim R = \dim R/\mathfrak{p}$, and two distinct vertices $\mathfrak{p}, \mathfrak{q}$ are joined by an edge if and only if $(\mathfrak{p}, \mathfrak{q})$ is an ideal of height one.

Next we are interested in the connectedness of $\mathbb{G}(R)$. That is characterized in the following statement. To this end we refer to the notion of connectedness in codimension one of $\text{Spec } R$ as defined by Hartshorne (see [6]).

Proposition 5.2. *Let (R, \mathfrak{m}) denote a local ring with $d = \dim R$. Then the following conditions are equivalent:*

- (i) *The graph $\mathbb{G}(R)$ is connected.*
- (ii) *$\text{Spec } R/0_d$ is connected in codimension one.*
- (iii) *For every ideal $JR/0_d$ of height at least two, $\text{Spec}(R/0_d) \setminus V(JR/0_d)$ is connected.*

Proof. By the definitions (see 5.1 and [6]) this is easily seen. See also [8, (3.6)]. \square

Next we describe when the endomorphism ring of $H_I^d(R)$, $d = \dim R$, is a local ring. We call an R -module X indecomposable if it is not the direct sum of two non-trivial submodules.

Theorem 5.3. *Let (R, \mathfrak{m}) denote a complete local ring and $d = \dim R$. For an ideal $I \subset R$ the following conditions are equivalent:*

- (i) *$H_I^d(R)$ is indecomposable.*
- (ii) *$T_I(R)$ is indecomposable.*
- (iii) *The endomorphism ring of $H_I^d(R)$ is a local ring.*
- (iv) *The graph $\mathbb{G}(R/Q_I(R))$ is connected.*

Proof. We may always assume that $Q = Q_I(R)$ is a proper ideal. In the case of $Q = R$ there is nothing to show. As it follows by above investigations we have the following isomorphisms

$$H_{\mathfrak{m}}^d(R/Q) \simeq H_I^d(R), K_{R/Q} \simeq T_I(R) \text{ and } \text{End } H_{\mathfrak{m}}^d(R/Q) \simeq \text{End } H_I^d(R),$$

where End denotes the endomorphism ring. That is, we have reduced the proof of the statement to the corresponding result for $H_{\mathfrak{m}}^d(R/Q)$. Note that $d = \dim R/Q$. Then the equivalence of the conditions is proved by Hochster and Huneke (see [8, (3.6)]). \square

Now we shall describe t , the number of connected components of $\mathbb{G}(R/Q_I(R))$.

Definition 5.4. Let I be an ideal in a local ring (R, \mathfrak{m}) . Suppose that $Q = Q_I(R)$ is a proper ideal. Let $G_i, i = 1, \dots, t$, denote the connected components of $G(R/Q)$. Let $Q_i, i = 1, \dots, t$, denote the intersection of all \mathfrak{p} -primary components of a reduced minimal primary decomposition of Q such that $\mathfrak{p} \in G_i$. Then $Q = \bigcap_{i=1}^t Q_i$ and $G(R/Q_i) = G_i, i = 1, \dots, t$, is connected. Moreover, let $I_i, i = 1, \dots, t$, denote the image of the ideal I in R/Q_i .

Theorem 5.5. Let I denote an ideal of a complete local ring (R, \mathfrak{m}) with $d = \dim R \geq 2$. Then

$$\text{End } H_I^d(R) \simeq \text{End } H_{I_1}^d(R/Q_1) \times \dots \times \text{End } H_{I_t}^d(R/Q_t)$$

is a semi-local ring, $\text{End } H_{I_i}^d(R/Q_i), i = 1, \dots, t$, is a local ring and therefore t is equal to the number of maximal ideals of $\text{End } H_I^d(R)$.

Proof. As in the proof in Theorem 5.3 we have $\text{End } H_{\mathfrak{m}}^d(R/Q) \simeq \text{End } H_I^d(R)$. For an integer $1 \leq i \leq t$ we define $\tilde{Q}_i = \bigcap_{j=1}^i Q_j$, in particular $\tilde{Q}_t = Q$. Then there is the short exact sequence

$$0 \rightarrow R/\tilde{Q}_{i+1} \rightarrow R/\tilde{Q}_i \oplus R/Q_{i+1} \rightarrow R/(\tilde{Q}_i + Q_{i+1}) \rightarrow 0.$$

Because G_{i+1} and G_j for $j = 1, \dots, i$, are not connected it follows by the definition that $\text{height}(\tilde{Q}_i + Q_{i+1}) \geq 2$ and therefore $\dim R/(\tilde{Q}_i + Q_{i+1}) \leq d - 2$. Whence the short exact sequence induces isomorphisms $H_I^d(R/\tilde{Q}_{i+1}) \simeq H_I^d(R/\tilde{Q}_i) \oplus H_I^d(R/Q_{i+1})$ and by induction

$$H_I^d(R/Q) \simeq \bigoplus_{i=1}^t H_I^d(R/Q_i).$$

Furthermore, because of Theorem 3.3 and Corollary 3.5 we have

$$H_I^d(R) \simeq H_{\mathfrak{m}}^d(R/Q) \text{ and } H_{I_i}^d(R/Q_i) \simeq H_I^d(R/Q_i) \simeq H_{\mathfrak{m}}^d(R/Q_i), i = 1, \dots, t.$$

Now by Matlis duality it turns out that

$$\text{End } H_I^d(R) \simeq \text{End } K_{R/Q} \text{ and } \text{Hom}_R(H_{\mathfrak{m}}^d(R/Q_j), H_{\mathfrak{m}}^d(R/Q_i)) \simeq \text{Hom}_R(K_{R/Q_i}, K_{R/Q_j})$$

for all $i, j = 1, \dots, t$. Moreover we see that $\text{Hom}_R(K_{R/Q_i}, K_{R/Q_j}) = 0$ for $i \neq j$ because

$$\text{Ass}_R \text{Hom}_R(K_{R/Q_i}, K_{R/Q_j}) = \text{Ass}_R K_{R/Q_j} \cap \text{Supp}_R R/Q_i = \emptyset$$

for all $i \neq j$ as follows by the definitions and Proposition 2.4. This implies the decomposition

$$\text{End } H_I^d(R) \simeq \text{End } H_{I_1}^d(R/Q_1) \times \dots \times \text{End } H_{I_t}^d(R/Q_t)$$

because $\text{End } K_{R/Q_i} \simeq \text{End } H_{I_i}^d(R/Q_i), i = 1, \dots, t$, as follows again by Matlis duality. By Theorem 5.3 the endomorphism ring of $H_{I_i}^d(R/Q_i), i = 1, \dots, t$, is a local ring. So we get the decomposition as a direct product of rings and $\text{End } H_I^d(R)$ is a semi-local ring with t as its number of maximal ideals. \square

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